Rotations on the Bloch Sphere

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The Bloch Sphere

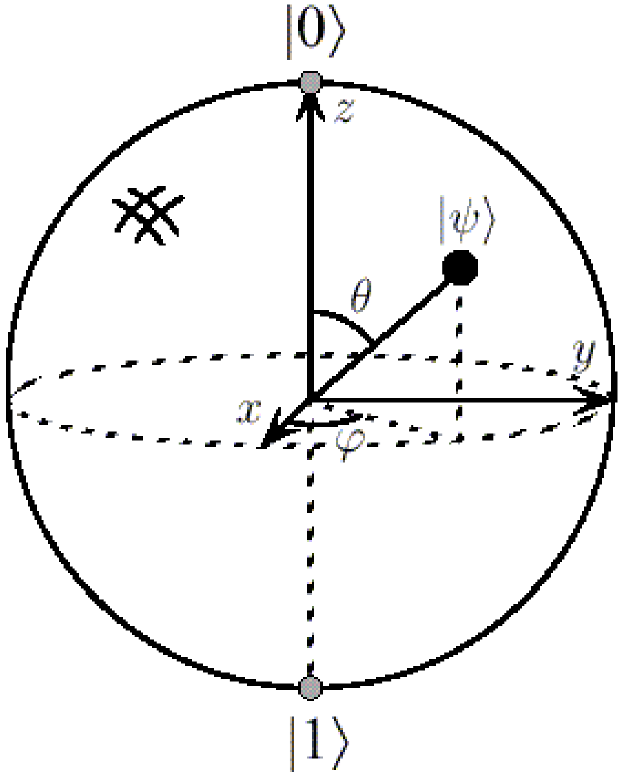
An arbitrary single qubit state can be written:

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, ϕ and γ are real numbers. The numbers $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ define a point on a unit three-dimensional sphere. This is the *Bloch sphere*. Qubit states with arbitrary values of γ are all represented by the same point on the Bloch sphere because the factor of $e^{i\gamma}$ has *no observable effects*, and we can therefore choose to write:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

The Bloch Sphere



The Density Operator

In order to relate unitary operations on a qubit state $|\psi\rangle$ to rotations on the Bloch sphere it turns out to be convenient to use the corresponding density operator ρ , defined as:

$$\rho = |\psi\rangle\langle\psi| = |\psi\rangle \otimes \langle\psi|$$

where

$$\langle\psi| = |\psi\rangle^\dagger$$

so

$$\rho = |\psi\rangle \otimes \langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

The Density Operator

By the definition of the outer product

$$\rho = \begin{pmatrix} \cos^2 \frac{\theta}{2} & e^{-i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ e^{i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} & \sin^2 \frac{\theta}{2} \end{pmatrix}$$

and using standard trigonometric identities

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta & \cos\phi \sin\theta - i \sin\phi \sin\theta \\ \cos\phi \sin\theta + i \sin\phi \sin\theta & 1 - \cos\theta \end{pmatrix}$$

The Density Operator

then grouping terms in the basis $\{I, X, Y, Z\}$, where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices, we have

$$\begin{aligned} \rho &= \frac{1}{2}(I + X\cos\phi\sin\theta + Y\sin\phi\sin\theta + Z\cos\theta) \\ &= \frac{1}{2}(I + \vec{\mathbf{r}}_\rho \cdot \vec{\sigma}) \end{aligned}$$

where I is the identity matrix, $\vec{\sigma}$ is the 3-element 'vector' of Pauli Matrices (X, Y, Z), and $\vec{\mathbf{r}}_\rho$ is the unit Bloch vector

$$\vec{\mathbf{r}}_\rho = (r_x,\ r_y,\ r_z) = (\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta)$$

Unitary Evolution of the Density Operator

Quantum circuits consist of combinations of quantum gates, each corresponding to a unitary operation on a qubit state. That is:

$$|\psi\rangle \mapsto U|\psi\rangle$$

where U is a unitary operator (matrix), i.e. $U^\dagger U = UU^\dagger = I$, so, recalling that the density operator $|\psi\rangle\langle\psi| = |\psi\rangle \otimes (|\psi\rangle)^\dagger$, it evolves as

$$\begin{aligned} |\psi\rangle\langle\psi| &\mapsto (U|\psi\rangle) \otimes (U|\psi\rangle)^\dagger \\ &\mapsto (U|\psi\rangle) \otimes (\langle\psi|U^\dagger) \\ &\mapsto U|\psi\rangle\langle\psi|U^\dagger \end{aligned}$$

Rotation Operators

The Pauli X, Y and Z matrices are so-called because when they are exponentiated, they give rise to the *rotation operators*, which rotate the Bloch vector $\vec{\mathbf{r}}_\rho$ about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes, by a given angle θ:

$$\begin{aligned} R_x(\theta) &\equiv e^{-i\frac{\theta}{2}X} \\ R_y(\theta) &\equiv e^{-i\frac{\theta}{2}Y} \\ R_z(\theta) &\equiv e^{-i\frac{\theta}{2}Z} \end{aligned}$$

Now, if operator A satisfies $A^2 = I$, it can be shown that

$$e^{i\theta A} = \cos(\theta)I + i\sin(\theta)A$$

Rotation Operators

And since the Pauli matrices satisfy $X^2 = Y^2 = Z^2 = I$, the rotation operators can be expanded as:

$$R_x(\theta) \equiv e^{-i\frac{\theta}{2}X} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\frac{\theta}{2}Y} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\frac{\theta}{2}Z} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Rotation about the $\hat{z}$ Axis

We can check that these operators do what they're supposed to by considering their action on the state ρ. For example, $R_z(\theta)$ evolves ρ to:

$$\begin{aligned}
\rho' &= R_z(\theta)\rho R_z(\theta)^\dagger \\
&= R_z(\theta)\ \frac{1}{2}(I + \vec{\mathbf{r}}_\rho \cdot \vec{\sigma})\ R_z(\theta)^\dagger \\
&= R_z(\theta)\ \frac{1}{2}(I + r_x X + r_y Y + r_z Z)\ R_z(\theta)^\dagger
\end{aligned}$$

It is easily verified that $R_z(\theta)$ is unitary, so $R_z(\theta)R_z(\theta)^\dagger = I$, and

$$\rho' = \frac{1}{2}(I + r_x R_z(\theta) X R_z(\theta)^\dagger + r_y R_z(\theta) Y R_z(\theta)^\dagger + r_z R_z(\theta) Z R_z(\theta)^\dagger)$$

Rotation about the $\hat{z}$ Axis

Now expand

$$\begin{aligned} R_z(\theta) X R_z(\theta)^\dagger &= \left(\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z\right) X \left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Z\right) \\ &= \cos^2\frac{\theta}{2} X + i\sin\frac{\theta}{2}\cos\frac{\theta}{2} XZ - i\sin\frac{\theta}{2}\cos\frac{\theta}{2} ZX \\ &\quad + \sin^2\frac{\theta}{2} ZXZ \end{aligned}$$

To evaluate this expression we use the algebra of the Pauli matrices.

Algebra of the Pauli Matrices

The algebra of the Pauli matrices can be summarised by the equation:

$$X^2 = Y^2 = Z^2 = -iXYZ = I$$

All the products of pairs of Pauli matrices can be calculated from the above equation.

Algebra of the Pauli Matrices

For example:

$$
\begin{aligned}
I &= -iXYZ \\
(I)Z &= (-iXYZ)Z \\
Z &= -iXY \\
(Z)Y &= (-iXY)Y \\
ZY &= -iX \\
(ZY)X &= (-iX)X \\
ZYX &= -iI \\
Z(ZYX) &= Z(-iI) \\
YX &= -iZ
\end{aligned}
$$

Algebra of the Pauli Matrices

The products of pairs of Pauli matrices are:

$$XY = -YX = iZ$$
$$YZ = -ZY = iX$$
$$ZX = -XZ = iY$$

which can be summarised as

$$\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k$$

where $\sigma_1 = X$, $\sigma_2 = Y$ and $\sigma_3 = Z$. Notice that non-identical Pauli matrices anticommute, i.e. $\sigma_i \sigma_j = -\sigma_j \sigma_i$ if $i \neq j$.

Rotation about the $\hat{z}$ Axis

We can now write

$$\begin{aligned}
R_z(\theta) X R_z(\theta)^\dagger &= \cos^2\frac{\theta}{2} X + i\sin\frac{\theta}{2}\cos\frac{\theta}{2} XZ - i\sin\frac{\theta}{2}\cos\frac{\theta}{2} ZX \\
&\quad + \sin^2\frac{\theta}{2} ZXZ \\
&= \cos^2\frac{\theta}{2} X + \sin\frac{\theta}{2}\cos\frac{\theta}{2} Y + \sin\frac{\theta}{2}\cos\frac{\theta}{2} Y - \sin^2\frac{\theta}{2} X \\
&= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) X + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, Y \\
&= \cos\theta\, X + \sin\theta\, Y
\end{aligned}$$

Rotation about the $\hat{z}$ Axis

Similarly we can show that

$$
\begin{aligned}
R_z(\theta) Y R_z(\theta)^\dagger &= \cos\theta\, Y - \sin\theta\, X \\
R_z(\theta) Z R_z(\theta)^\dagger &= Z
\end{aligned}
$$

So we can now rewrite

$$
\begin{aligned}
\rho' &= \frac{1}{2}(I + r_x R_z(\theta) X R_z(\theta)^\dagger + r_y R_z(\theta) Y R_z(\theta)^\dagger + r_z R_z(\theta) Z R_z(\theta)^\dagger) \\
&= \frac{1}{2}(I + r_x(\cos\theta\, X + \sin\theta\, Y) + r_y(\cos\theta\, Y - \sin\theta\, X) + r_z Z) \\
&= \frac{1}{2}(I + (r_x\cos\theta - r_y\sin\theta)X + (r_x\sin\theta + r_y\cos\theta)Y + r_z Z)
\end{aligned}
$$

Rotation about the $\hat{z}$ Axis

and recalling that we can write

$$
\begin{aligned}
\rho' &= \frac{1}{2}(I + \vec{\mathbf{r}}_{\rho'} \cdot \vec{\sigma}) \\
&= \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)
\end{aligned}
$$

we can see that

$$
\begin{aligned}
r'_x &= r_x \cos\theta - r_y \sin\theta \\
r'_y &= r_x \sin\theta + r_y \cos\theta \\
r'_z &= r_z
\end{aligned}
$$

Rotation about the $\hat{z}$ Axis

so the Bloch vector of the new state is

$$\vec{\mathbf{r}}_{\rho'} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathbf{r}}_{\rho}$$

and the matrix is the usual 3D-rotation matrix for a rotation about the $\hat{z}$ axis by and angle of θ, as required. We can similarly show that $R_x(\theta)$ and $R_y(\theta)$ perform rotations of the Bloch vector about the x and y axes by an angle θ.

Rotation About an Arbitrary Axis

We can use the rotation operators about the y and z axes to construct the operator for rotation by and angle α about an arbitrary axis $\hat{n}$, since we can decompose it as:

$$\begin{aligned} R_{\hat{n}}(\alpha) &= R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi) \\ &= R_z(\phi)R_y(\theta)R_z(\alpha)R_y(\theta)^\dagger R_z(\phi)^\dagger \end{aligned}$$

Taking the rotations one by one, $R_z(-\phi)$ first rotates $\hat{n}$ into the x-z plane, $R_y(-\theta)$ then rotates it into the z axis, $R_z(\alpha)$ performs the desired rotation about $\hat{n}$, and $R_y(\theta)$ and $R_z(\phi)$ rotate $\hat{n}$ back to its original orientation.

Rotation About an Arbitrary Axis

Working from the inside out, we can rewrite

$$\begin{aligned} R_{\hat{n}}(\alpha) &= R_z(\phi)R_y(\theta)R_z(\alpha)R_y(\theta)^\dagger R_z(\phi)^\dagger \\ &= R_z(\phi)R_y(\theta)\left[\cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}Z\right]R_y(\theta)^\dagger R_z(\phi)^\dagger \end{aligned}$$

And using the Pauli matrix algebra as earlier, we can show that

$$R_y(\theta)ZR_y(\theta)^\dagger = \cos\theta\ Z + \sin\theta\ X$$

and $R_y(\theta)R_y(\theta)^\dagger = I$, so

$$R_{\hat{n}}(\alpha) = R_z(\phi)\left[\cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}(\cos\theta\ Z + \sin\theta\ X)\right]R_z(\phi)^\dagger$$

Rotation About an Arbitrary Axis

$$R_{\hat{n}}(\alpha) \quad = \quad R_z(\phi)\left[\cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}(\cos\theta\ Z + \sin\theta\ X)\right]R_z(\phi)^\dagger$$

and using $R_z(\theta)ZR_z(\theta)^\dagger = Z$ and $R_z(\theta)XR_z(\theta)^\dagger = \cos\theta\ X + \sin\theta\ Y$ we obtain

$$\begin{aligned}
R_{\hat{n}}(\alpha) &= \cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}(\cos\theta\ Z + \sin\theta\ [\cos\phi\ X + \sin\phi\ Y]) \\
&= \cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}(\sin\theta\cos\phi\ X + \sin\theta\sin\phi\ Y + \cos\theta\ Z) \\
&= \cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}\ (n_xX + n_yY + n_zZ) \\
&= \cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}\ \hat{n}\cdot\vec{\sigma}
\end{aligned}$$

Rotation About an Arbitrary Axis

We can rewrite this result as an operator exponential, because

$$\begin{aligned}
(\hat{n}\cdot\vec{\sigma})^2 &= (n_x X + n_y Y + n_z Z)(n_x X + n_y Y + n_z Z) \\
&= n_x^2 X^2 + n_x n_y XY + n_x n_z XZ + \\
&\quad n_y n_x YX + n_y^2 Y^2 + n_y n_z YZ + \\
&\quad n_z n_x ZX + n_z n_y ZY + n_z^2 Z^2 \\
&= (n_x^2 + n_y^2 + n_z^2)\, I \\
&= I
\end{aligned}$$

since $\hat{n}$ is a unit vector, so we can use $A = \hat{n}\cdot\vec{\sigma}$ in the identity

$$e^{i\theta A} = \cos(\theta) I + i \sin(\theta) A$$

Rotation About an Arbitrary Axis

and we can write

$$
\begin{aligned}
R_{\hat{n}}(\alpha) &= \cos\frac{\alpha}{2} I - i \sin\frac{\alpha}{2}\ \hat{n}\cdot\vec{\sigma} \\
&= \exp(-i\frac{\alpha}{2}\ \hat{n}\cdot\vec{\sigma})
\end{aligned}
$$

which is our final result for the operator to transform a state ρ such that its Bloch vector $\vec{\mathbf{r}}_\rho$ is rotated about the $\hat{n}$ axis by an angle α, i.e.

$$\rho' = R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger$$

or equivalently

$$|\psi'\rangle = R_{\hat{n}}(\alpha)|\psi\rangle$$

Arbitrary Unitary Operator

Since $R_{\hat{n}}(\alpha)$ can rotate a Bloch vector into any other Bloch vector, which includes all possible qubit states up to a global phase factor, an arbitrary single qubit unitary operator can be written in the form:

$$U = \exp(i\gamma) R_{\hat{n}}(\alpha)$$

for some real numbers γ and α and a real three-dimensional unit vector $\hat{n}$. For example, consider $\gamma = \pi/2$, $\alpha = \pi$, and $\hat{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

$$\begin{aligned} U &= \exp(i\pi/2)\left[\cos\left(\frac{\pi}{2}\right) I - i \sin\left(\frac{\pi}{2}\right) \frac{1}{\sqrt{2}}(X + Z)\right] \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

which is the Hadamard gate H.

Future Topics

- Generalisation of the Bloch sphere to mixed states
- Generalizations to more qubits